Journal of Namibian Studies, 33 S2(2023): 5320-5326 ISSN: 2197-5523 (online)

Perturbation Quadratic Forms Of L^P Contraction Semigroup

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Abstract

A complex matrix is associated with a sequence of real numbers. The present study demonstrates the methodology for computing the actual perturbation values and highlights their resemblance to the singular values in terms of various properties. Real perturbation values can be utilized to compute the real pseudospectra and real stability radii. The primary outcome pertains to the signature exhibited by real quadratic forms within complex vector spaces. In this work I showed a result of the contraction semigroup L^P norm for vector-valued functions, the Markovian semigroup which is symmetric and not the only one on L^2 contraction semigroup but also a L^P contraction of semigroup for any $P \in [1,\infty)$ and I prove that the sharp estimate of the defective logarithmic Sobolev inequality holds for $t > 0$ *and* $1 < p \leq 2$

Key Words: (DLS) The Defective Logarithmic Sobolev inequality, quadratic forms, comparison theorem, a contraction Markovian semigroup, Trotter product formula, Hausdorff-Young inequality.

Introduction: Discussing the perturbed as semigroups. To Let $\{T_t\}$ and $\{T_t^\circ\}$ be semigroup, the assumption that is basic in its type is that $\{T_t\}$ and satisfies hyperboundedness. I will treat with that semigroup $\{T_t\}$ to act on the vector-valued functions. As well as supposing that areal value of function V and $pL(k)$ - value of function R are given. $pL(k)$ is operator norm, that be denoted by $\|\cdot\|_{op}$ I always assume that V and R are measurable.

Considering the quadratic forms:

$$E^{V}(f,g) = E(f,g) + \int_X Vf\,\overline{g}\,dm \;..............(1)$$

And

$$E^{\circ R}(u,v) = E^{\circ}(u,v) + \int_X (\mathrm{Ru},\mathrm{v})_k\,dm \;..............(2)$$

Where E and E° are forms (quadratic) that are associated with semigroups $\{T_t\}$ and $\{T_t^{\circ}\}$ respectively.

Considering the case which were V and R are readily bound. So, E^{V} and $E^{\circ R}$ are totally closed as quadratic forms to same domains as E and E°.

Comparison Theorem: Allowing (X,β) to be a measured space and m be $\sigma-$finite a measure. Supposing to give a strongly continuous "contraction" semigroup $\{T_t\}$ on L_t^2. I supposing; T_t is Markovian, i.e., if $f \in L^2$ satisfiers $0 \le f \le 1$, then $0 \le T_t f \le 1$. Here these inequality hold a.e. The generator $\{T_t\}$ is denoted by A and its resolvents $G_\alpha, \alpha > 0$. Besides, I assume that: $\int_X T_t f dm \le \int_X f dm$, $f \in L^1 \cap L_+^2$, where + stand for the non-negative function. Note that it $\{T_t\}$ is symmetric; the above property follows from the Markovian property. Thus $\{T_t\}$ becomes semigroup which is known as contraction, on L^1. The Ries-Thorin interpolation theorem $\{T_t\}$ is a contraction semigroup on L^P for $P \in \{1,\infty\}$. To be precise, for $f \in L^2 \cap L^P$, $T_t f \in L^2 \cap L^P$ and: $\|T_t f\|_P \le \|f\|_P$. Therefore, $P \in [1,\infty)$, T_t be extended to abounded linear operator L^P

But, for $P = \infty$, T_t is denoted only on $\overline{L^2 \cap L^\infty}^{\|\cdot\|_\infty}$, the completion of $L^2 \cap L^\infty$ concerning $\|\cdot\|_\infty$.

2- The main results:

Proposition 2.1: Assume that V and R are bounded and :

$$V(x)|k|^2 \leq (R(x)k,k)_k \quad ,\forall k \in k^* \quad \text{a.e. } x \text{ ....(3)}$$

Then T_t^v is a positivity preserving semigroup and:

$$|T_2^{\circ R} u| \leq T_t^v |u| \qquad \text{.............(4)}$$

Proof: Observing that

$E^V(|f|,|f|) \leq E^V(f,f)$ in additions, T_t^v is a positive.

Proving (4). using Trotter product equation,

Having: $T_t^v = \lim_{n\to\infty}(e^{-tV/n}T_{t/n})^n f$

Then: $T_t^{\circ R} u = \lim_{n\to\infty}(e^{tR/n}T_{t/n})^n u$

So: $T_t^{\circ} u \leq T_t |u|,$

As well as $\|e^{-tR(x)}\|_{op} \leq e^{-tv(x)}$.

I have

$$\|T_t^{\circ R} u\| = \lim_{n\to\infty}|(e^{-tR/n}T_{t/n}^{\circ})^n u| \leq \lim_{n\to\infty}(e^{-tv/n}T_{t/n})^n |u| = T_t^v |u|$$

This completes the proof.

Assuming a condition (DLS) (the defective logarithmic as Sobolev inequality).

(DLS) There as present $a > 0, \beta \geq 0$:

$$\int_X f^2 Log\left(|u|^2/\|u\|_2^2\right)dm \leq \alpha E^*(u,u) + \beta\|f\|_2^2 \text{ for } u \in Dom(E^*)$$

Having inequality to E° :

$$\int_X |u|^2 Log\left(|u|^2/\|u\|_2^*\right)dm \leq \alpha E^*(u,u) + \beta\|u\|_2^2 \text{ fo} u \in Dom(E^*)$$

With assumption, $t \geq 0$ and $I < p < q < \infty$ with

$$(q-1)/(p-1) \le e^{4t/\alpha},$$

$$\|T_t\|_{p\to q} \le \exp\{\beta(1/p - 1/p)\} \qquad \text{...........(7)}$$

Proposition 2.2: Assuming (DLS) holds.

$t > 0$ and $I < p < q < \infty$ such that $(q-1)/(q-1) \le e^{4t/\alpha'}$ with $\alpha' > \alpha$,

Having:

$$\|T_t^V\|_{p\to q} \le \exp\{\beta(1/p - 1/q)\}\|e^{-V}\|_r^t \, e^{st} \qquad \text{...........(8)}$$

And,

$$r = \frac{1}{4}\{1/\alpha - 1/\alpha'\}^{-1}\{p^2/p-1 \vee q^2/q-1\}, \; S = \frac{4\beta}{\alpha}$$

Where $\vee$ can denote its maximum, particularly, for which $t < 0$

$$\|T_t\|_{p\to q} \le \|e^{-v}\|^t_{\alpha p^2/4(p-1)} \; e^{4\beta t/\alpha} \qquad \text{...........(9)}$$

Corollary 2.3: assuming (DLS) holds. Then for $t > 0$ and $1 < p \le 2$ such that $e^{4t/\alpha'} \ge 1$ with $\alpha' < \alpha$, I have :

$$\|T_t^V\|_{2p} \le e^{\frac{4\beta}{\alpha}t}\|e^{-v}\|^t_{p-1}$$

Where $V = \dfrac{\alpha.\alpha'}{\alpha - \alpha'}$

Proof: Take $e^{4t/\alpha''}$ Define a monotonic sequence

$$\{p_n\}_{n=0}^{\infty}, \; 1 < p_n \le 2 \;, \;\; \text{where} \quad \frac{1}{p_{n+1}} = \frac{1}{r_n} + \frac{1}{p_n}$$

Proposition (2) gives : $e^{4t/n\alpha} = \dfrac{1}{p_n - 1}$

$$\|e^{-tV/n}T_{t/n}f\|_{p_{n+1}} \le \|e^{tV/n}\|_{r_n}\|T_{t/n}f\|_{p_n} \le \|e^{-tV/n}\|_{r_n}\|f\|_{p_n}$$

Taking the supremum over the norm and hence

Consequently, I have :

$$\left\|\left(e^{-tV/n}T_{t/n}\right)\right\|_{p_{n+1}} \leq \left\|e^{tV/n}T_{t/n}\right\|_{p_n}^{n}$$

Letting $n \to \infty$ I have :

$$\left\|T_t^V\right\|_p \leq \left\|e^{-v}\right\|_{p-1}^{t} e^{st} \quad \text{where} \quad s = 4\beta/\alpha$$

Proposition 2.4: Suggesting as exists $\overset{\square}{\alpha} > \alpha$ such that $\exp\left\{\overset{\square}{\alpha}\|R\|_{op}\right\} \in L^1$ And:

$$E^{\circ R}(u,v) = E(u,v) + \int_X (Ru,v)_k \, dm.$$

Is well defined for $u,v \in Dom(E^{\circ})$ as a lower semibounded closed quadratic in $L^2(m)$.

Moreover if $D \subseteq Dom(E^{\circ})$ is dense in $Dom(E^{\circ})$, then it is dense $Dom(E^{\circ R})$ also.

Proof: By the Hausdorff-young inequality $st \leq s\ Log\ s - s + e^t, s > 0, t \in R$,

For

$u \subseteq Dom(E^{\circ})$ with $\|u\|_2 = 1$

$$\int_X \left|(Ru,u)_k\right| dm \leq \int_X \|R\|_{op} |u|^2 \, dm$$

$$= \frac{1}{\overset{\square}{\alpha}} \int_X |u|^2 \left(\overset{\square}{\alpha}\|R\|_{op}\right) dm$$

$$= \frac{1}{\overset{\square}{\alpha}} \int_X \left\{|u|^2 Log|u|^2 - |u|^2 + \exp\left(\overset{\square}{\alpha}\|R\|_{op}\right)\right\} dm$$

$$= \frac{1}{\overset{\square}{\alpha}} \alpha E^{\circ}(u,u) + \beta - 1 + \left\|\exp\left\{\overset{\square}{\alpha}\|R\|_{op}\right\}\right\|_1 dm$$

For general u having :

$$\int_X \left|(Ru,u)_k\right| d1m = \frac{\alpha}{\overset{\square}{\alpha}} E^{\circ}(u,u) + \frac{1}{\overset{\square}{\alpha}}\left(\beta + 1 + \left\|\exp\left\{\overset{\square}{\alpha}\|R\|_{op}\right\}\right\|_1\right)\|u\|_2^2$$

Denoting the semigroups that are associated with $E^{\circ R}$ and E^{V} by $\{T_t^{\circ R}\}$ and $\{T_t^{V}\}$. To have following comparison of theorem.

Theorem 2.5: Having :

$$\left|T_t^{\circ R} u\right| \leq T_t^{V} \left|u\right| \qquad \text{............} \ (10)$$

Proof: By proposition (2.1), to have:

$$\left|T_t^{\circ R_n}\right| \leq T_t^{V_n} \left|u\right|$$

By letting, I can get the desired result.

This theorem proof by an approximation to set

$$X_n = \left\{x \in X ; \left\|R(x)\right\|_{op} \leq n, \left|V(x)\right| \leq n\right\}$$

And define: $R_n = \big|_{x_n} R,\ V_n = \big|_{x_n} V$.

Then it will be shown as the convergence of $E^{\circ R_n}$. to use convergence of quadratic forms. As well as saying the quadratic forms sequence of E^{n} converges to E if:

(M.1) For any sequence $\{u_n\}$ to u weakly, for that :

$$E(u,u) \leq \underline{\lim}_{n \to \infty} E^{n}(u_n, u_n) \qquad \text{............}(11)$$

(M.2) For any $u \in H$, exist a sequence $\{u_n\}$ to u strongly that:

$$E(u,u) \leq \overline{\lim}_{n \to \infty} E^{n}(u_n, u_n) \qquad \text{............}(12)$$

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